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SURFACE PATTERN FORMATION FOR ONE-DIMENSIONAL INTERFACES

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1. INTRODUCTION

The pattern formation is ubiquitous in nature and plays an important role in the technical sciences, because it is one of the most fascinating features of non-equilibrium systems [1].

Typical examples can be found in many startlingly similar interfaces which emerge in very different processes, such as growth of amorphous [2] and epitaxial thin films [8], or erosion by ion beam sputtering (IBS) [10]. These pattern-forming surfaces can be classified into different categories according to the stationary or time-dependent (coarsening) behavior of the typical pattern length scale l [7].

A one-dimensional counterexample of an interface equation leading to stationary pattern with constant wavelength and amplitude would be really amazing, because it is supported our understanding of interrupted coarsening. A continuum two-dimensional model has been introduced that describes interesting (sub) micrometric features of surfaces eroded by IBS in [3], [6].

Moreover, the molecular beam epitaxy (MBE) is important in many applications, e.g., in the electronics industry a thin film of a solid semiconductor material needs to be deposited on a solid semiconductor substrate. Very often the growing film does not remain planar during its growth and develops various kinds of surface structures. The types of these structures depend on physical characteristics of the materials as well as on the growth conditions [7].

The aim of this paper is to investigate analytically a one-dimensional conserved Kuramoto-Sivashinsky equation (CKS) following the work of Muñoz-Garcia, Cuerno and Castro [7].

2. PATTERN FORMATION

The lowest order near threshold of the morphological instability can be described the following one-dimensional equation for the evolution of the surface height

$$u_t(x,t) = -\nu u_{xx} - K u_{xxxx} + \lambda_1 (u_x)^2 - \lambda_2 \left((u_x)^2\right)_{xx}, \tag{1}$$

where u is the height of the bombarded surface, and the parameters ν, K, λ_1 and λ_2 depends on the experimental conditions. According to [7], the deterministic equation (1) has not been systematically studied. They want to determine a long-wavelength instability, hence they have been restricted themselves to positive values of ν and K. Moreover, λ_1 and λ_2 are important parameters, which can be determine as shown in [4]. So, these parameters are considered positive values. In that case, when $\lambda_2 = 0$, equation (1) reduces to the well known Kuramoto-Sivashinsky (KS) equation which is a paradigm of spatiotemporal chaos [5[, [9]. Its nonlinear description stabilizes the system and a (disordered) pattern develops that is characterized by a wavelength that does not coarsen, and by chaotic cell dynamics. On large length scales, the KS system can be effectively described by the stochastic Kardar-Parisi-Zhang (KPZ) equation [19], paradigmatic of kinetic roughening. In particular, the surface roughness (global RMS width, $W = w(x_0 = L)$, $w^2(x_0) = \sum_x \left[u(x,t) - \bar{u}_{x_0}\right]^2 / x_0$, where $\bar{u}_{x_0}$ denotes the mean value of u at x_0) [20] for a KPZ interface scales as a power law with the lateral system size L. On the other hand, for $\lambda_2 \neq 0$ and $\lambda_1 = 0$, Equation (1) reduces to the CKS equation. This equation has been investigated in the context of amorphous thin film growth and step dynamics on vicinal surfaces, in this case, the linear instability evolves into an ordered pattern of paraboloids with uninterrupted coarsening [3].

3. CONSERVATIVE KURAMOTO-SIVASHINSKY EQUATION (CKS)

Our aim is to investigate one-dimensional problems. We can describe the evolution of surface with Kuramoto-Shivashinsky equation applying $\lambda_2 = 0$ in equation (1)

$$u_t(x,t) = -\nu u_{xx} - K u_{xxxx} + \lambda_1 (u_x)^2 .$$

Another special case of (1) is the Conserved Kuramoto-Sivashinsky equation (CKS), which is derived from (1) using $\lambda_1 = 0$

$$u_t(x,t) = -\nu u_{xx} - K u_{xxxx} - \lambda_2 \left((u_x)^2\right)_{xx} . \tag{2}$$

We can reduce the number of parameters applying a scaling for x and t with $(K/\nu)^{1/2}$ and K/ν^2, respectively, and rearranging (2) in the form

$$u_t + u_{xx} + u_{xxxx} + \gamma\left((u_x)^2\right)_{xx} = 0 ,$$

where $\gamma = \lambda_2 / K$.

Our aim is to examine solution of the form

$$u(t,x) = u(t, \lambda(t) + u)$$

with some function $\lambda(t)$.

Therefore, we introduce the similarity form for u as

$$u(t,x) = t^{\alpha} f\left(x, t^{-\beta}\right), \text{ with } \quad \lambda(t) = ct^{\beta} \ , \tag{3}$$

where α represents the velocity of arrival to the surface, β denotes the velocity in surface direction and $\gamma = 1/\beta$ denotes the coarsening.

We shall consider (2) as

$$u_t + \frac{\partial}{\partial x^2}\left[u + u_{xx} + \gamma (u_x)^2\right] = 0 \tag{4}$$

Applying similarity variable $\eta = xt^{-\beta}$ for equation (4) one gets

$$t^{\alpha-1}(\alpha f - \beta \eta f') + t^{\alpha-2\beta} f'' + t^{\alpha-4\beta} f^{(4)} + \gamma t^{2(\alpha-\beta)-2\beta}\left(f'^2\right)'' = 0. \tag{5}$$

Equating the exponents of t at the left side and with the choice of $\alpha = 1$ and $\beta = 1/2$ we can rewrite (5) in the form

$$f - \frac{1}{2}\eta f' + f'' + t^{-1} f^{(4)} + \gamma \left(f'^2\right)'' = 0. \tag{6}$$

Remark that at $t = 0$ we can eliminate u_{xxxx}. In general, there are no similar solutions of the form (3).

If $t \to \infty$, then we suppose that $t^{-1} f^{(4)} \to 0$. Assuming that $f^{(4)} \equiv 0$ then we obtain f in the form $f(\eta) = a\eta^3 + b\eta^2 + c\eta + e$. To find even solution supposing that $f(\eta) = f(-\eta)$ to equation (6) we get for parameters a and c, that $a = 0$, $c = 0$ and we obtain solution u as follows

$$u(x,t)=t\left(b\frac{x^2}{t}+e\right),$$

with some parameters b and e.
From this, we can formulate the solution as

$$u(x,t)=e\left(t+\frac{b}{e}x^2\right)_+ . \qquad (7)$$

Multiplying equation (4) with u and integrating twice one gets

$$\frac{d}{dt}\frac{1}{2}\int_R u^2 = \int_R (u_x)^2 - \int_R (u_{xx})^2 . \qquad (8)$$

Substituting u of the form (7) into CKS differential equation (4) one gets that

$$\frac{b}{e}=-\frac{1}{4\gamma} .$$

Therefore, we want to find positive even solution

$$u(x,t)=\begin{cases}-\dfrac{1}{4\gamma}x^2 & \text{if } |x|<y(t)\\ 0 & \text{if } |x|\geq y(t)\end{cases}$$

to equation (4), where $y(t)$ is an unknown function. Remark, that height u is invariant under global height shifts $u(x,t)\to u(x,t)+const.$ [7].
Substituting u into equation (8) we have

$$\frac{d}{dt}\int_0^{y(t)}\frac{1}{8}x^4dx=\int_0^{y(t)}x^2dx-\int_0^{y(t)}dx .$$

Taking the integrals it reduces to the first order ordinary differential equation

$$\frac{1}{8} y^3(t) y'(t) = \frac{1}{3} y^2(t) - 1 .$$

The solution to this differential equation can be given by

$$t = \frac{9}{16}\left[\frac{1}{3} y^2(t) - \frac{1}{3} y^2(0) - \ln\left|\frac{\frac{1}{3} y^2(t) - 1}{\frac{1}{3} y^2(0) - 1}\right|\right].$$

From that expression we obtain the values of T depending on initial value $y(0)$ where u approaches to 0 (see Fig.1.)

$$T = \frac{9}{16}\left[\ln\left|\frac{1}{3} y^2(0) - 1\right| - \frac{1}{3} y^2(0)\right].$$

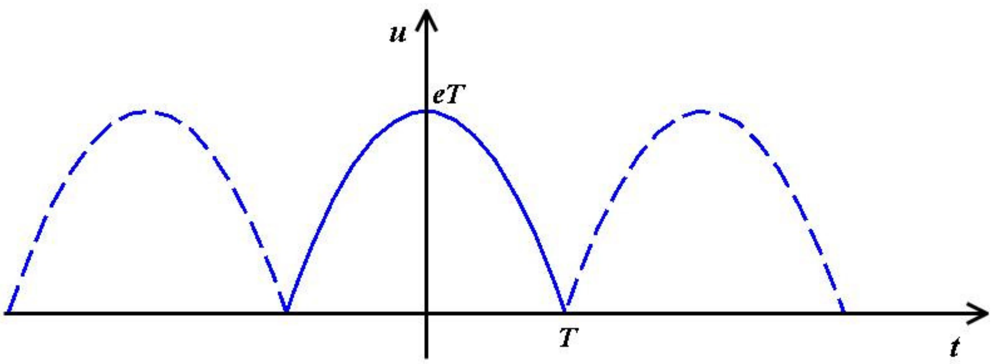

Figure 1. Similarity solution $u(t)$

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